Journal of Statistical Physics, Vol. 113, Nos. 3/4, November 2003 (© 2003)

Harmonic Interaction Model and Its Applications in Bose–Einstein Condensation

Jun Yan[1]

Received April 11, 2003; accepted June 10, 2003

We exactly solve the model of N harmonic interacting Bosons in a harmonic trap in any dimension. The exact ground state wavefunction, free energy, spectrum, and low excitation states are calculated. The finite particle number effect is addressed when the exact solution is compared with a mean field solution. Then we compare the harmonic interaction system with a pseudo-potential interaction system. In spite of the seemingly quite different nature of interaction, several similarities are found between the two systems.

KEY WORDS: Bose–Einstein condensation; harmonic interaction model; exact solution; model comparison.

1. INTRODUCTION

Since the breakthroughs$^{(1,2)}$ on Bose–Einstein Condensation (BEC) in a dilute alkali gas in 1995, there has been a gold rush in the study of dilute Bose gases in confined systems. The properties of ground state, collective modes, and vortices under rotation, etc. have been extensively studied both experimentally and theoretically. The dominant methods in current theoretical BEC studies are the mean field type GP equations, where the short-range interaction between particles is modeled by a pseudo-potential determined by scattering length. There are two natural questions one would like to ask. First, how good is the mean-field approximation? The BEC system realized in the experiments is usually mesoscopic (particles number $10^4 \sim 10^7$). As the experimental techniques improve, there have been

---

[1] Department of Physics, New York University, 4 Washington Place, New York, New York 10003; e-mail: jy272@nyu.edu

0022-4715/03/1100-0623/0 © 2003 Plenum Publishing Corporation

increasing demands for the investigation of the finite number effect, which goes beyond the scope of mean field approximation. Second, does the nature of interaction matter at all? If the answer is no, one can learn a great deal of one system from another. In this paper, both questions are tentatively addressed by inspection of the harmonic interaction model. In Section 2, the model is exactly solved and compared with mean field solution. In Section 3, we further compare it with the pseudo-potential system.

2. EXACT SOLUTION OF HARMONIC INTERACTION MODEL

For quantum Bose systems, exactly solvable models are very rare and often limited to one dimension. In 1960, Girardeau(3) showed that the 1-d point hard core Bose system can be exactly mapped onto the 1-d ideal Fermion gas. In 1963, Lieb *et al.*(4) solved the 1-d δ-interacting uniform Bose system using the Bethe ansatz. Recently, there have been works(5, 6) on Richardson's pairing model for a Bose system. Harmonic interaction model is one which can be solved in any dimension for any particle number. It describes N Boson particles with harmonic pair interaction in an external harmonic trap. For simplicity, we use an isotropic harmonic trap $u(\mathbf{r}) = \frac{1}{2} m\omega_0 r^2$ and set $m = \hbar = \omega_0 = 1$. The pair interaction can be anisotropic but we will focus on the isotropic case in this paper. Its Hamiltonian can be written as

$$H_{\text{har}} = \frac{1}{2} \sum_{i=1}^{N} (\mathbf{p}_i^2 + \mathbf{r}_i^2) + K \sum_{i<j} (\mathbf{r}_i - \mathbf{r}_j)^2 \tag{1}$$

where K is the strength of the interaction. This model was first used in nuclear physics to model the strong nuclear interaction, then proposed by Katzper and Percus(7) in 1967 to study the helium atom and later restudied by Wilkin *et al.*(8) and Zaluska-Kotur *et al.*(9) for the Bose system.

Equation (1) in d-dimensions is the sum of d one dimensional harmonic interaction Hamiltonians, so it is sufficient to study the one dimensional model. Now the Hamiltonian becomes

$$H_{\text{har}} = \frac{1}{2} \sum_{i=1}^{N} (p_i^2 + x_i^2) + K \sum_{i<j} (x_i - x_j)^2 \tag{2}$$

Introducing the center of mass and relative coordinates and momenta: $x_c = \frac{1}{N} \sum_{i=1}^{N} x_i$, $p_c = \frac{1}{N} \sum_{i=1}^{N} p_i$, and $x_i' = x_i - x_c$, $p_i' = p_i - p_c$ $(i = 1,\ldots, N)$, the Hamiltonian can be separated into the relative and center of mass parts:

$$H_{\text{har}}(\omega) = \frac{1}{2}\sum_{i=1}^{N} (p_i'^2 + \omega^2 x_i'^2) + \frac{N}{2}(p_c^2 + x_c^2)$$
$$= H_{\text{rel}}(\omega) + H_{\text{com}} \tag{3}$$

where $\omega = \sqrt{1+2NK}$.

We have the following commutation relations:

$$[x_c, p_c] = i/N \tag{4}$$
$$[x_j', p_k'] = i\alpha_{jk} \tag{5}$$
$$[x_c, p_j'] = 0 \qquad [x_i', p_c] = 0 \tag{6}$$

where $\alpha_{jk} = \delta_{jk} - 1/N$. From (6), we can see that $H_{\text{rel}}(\omega)$ and H_{com} are independent and the N-body wavefunction can also be written as the product of relative and center of mass parts:

$$\Psi(x_1,\ldots, x_N; \omega) = \psi_{\text{rel}}(x_1',\ldots, x_N'; \omega)\, \phi_{\text{com}}(x_c) \tag{7}$$

If we can solve $H_{\text{rel}}(\omega)\,\psi_{\text{rel}}(\omega) = E_{\text{rel}}(\omega)\,\psi_{\text{rel}}(\omega)$ and $H_{\text{com}}\phi_{\text{com}} = E_{\text{com}}\phi_{\text{com}}$ separately, then $E = E_{\text{rel}}(\omega) + E_{\text{com}}$ will be the total energy.

Equation (4) tells us that the center of mass is just a single harmonic oscillator. The eigen-wavefunction is

$$\phi_{\text{com}, m} = a_c^{\dagger\, m}\phi_0(\sqrt{N}\, x_c) \tag{8}$$

with corresponding energy

$$E_{\text{com}, m} = m + \tfrac{1}{2} \qquad (m = 0, 1,\ldots) \tag{9}$$

where a_c and $a_c^\dagger$ are annihilation and creation operators of center of mass satisfying $[a_c, a_c^\dagger] = 1/N$ and ϕ_0 is the ground state 1-d single harmonic oscillator eigen-wavefunction.

Since $\alpha_{jk} \neq \delta_{jk}$ in (6), the motions of N relative coordinates are not independent to each other. We can not just replace the ordinary coordinates in the N non-interacting Boson harmonic oscillators wavefunction by relative coordinates.

Define the relative annihilation and creation operators:

$$\begin{cases} a_j' = \dfrac{1}{\sqrt{2\omega}}(\omega x_j' + ip_j') \\[2ex] a_j'^{\,\dagger} = \dfrac{1}{\sqrt{2\omega}}(\omega x_j' - ip_j') \end{cases} \tag{10}$$

They satisfy

$$[a'_j, a'^{\dagger}_k] = \alpha_{jk} \tag{11}$$

Then the relative Hamiltonian can be written as

$$H_{\text{rel}}(\omega) = \omega \sum_{i=1}^{N} a'^{\dagger}_i a'_i + \frac{1}{2}(N-1)\,\omega \tag{12}$$

The ground state of $H_{\text{rel}}(\omega)$ satisfies $a'_i \psi_{\text{rel},0} = 0$ for all i's. It is easy to construct an unnormalized wavefunction

$$\psi_{\text{rel},0} = \exp\left(-\frac{\omega}{2} \sum_{i=1}^{N} x'^{2}_i\right) \tag{13}$$

with ground state energy $E_{\text{rel},0} = \frac{1}{2}(N-1)\,\omega$. From (2), the normalized ground state Eq. (7) is

$$\Psi_0(x_1,..., x_N) = \frac{\omega^{(N-1)/4}}{\pi^{N/4}} \exp\left[-\frac{1}{2}\omega \sum_{i=1}^{N} x'^{2}_i - \frac{1}{2} N x_c^2\right] \tag{14}$$

The excited states can be constructed based on

$$[H_{\text{rel}}(\omega), a'^{\dagger}_i] = \omega \sum_{j=1}^{N} a'^{\dagger}_j \alpha_{ji} = \omega a'^{\dagger}_i \tag{15}$$

where we have used $\sum_{j=1}^{N} a'^{\dagger}_j = 0$. So if $\psi_{\text{rel},0}$ is the ground state, $a'^{\dagger n}_j \psi_{\text{rel},0}$ will be the unsymmetrized eigenstate with energy $E_{\text{rel},0} + n\omega$. This immediately suggests the general form of symmetrized eigen-wavefunction is

$$\psi_{\text{rel},\{n_i\}} = \sum_P P \prod_{i=1}^{N} a'^{\dagger n_i}_i \psi_{\text{rel},0} \tag{16}$$

where P means all permutations between i's with different n_i's. The corresponding energy

$$E_{\text{rel},\{n_i\}} = E_{\text{rel},0} + \omega \sum_{i=1}^{N} n_i \tag{17}$$

The solution (16) looks exactly the same as for N non-interacting Bose harmonic oscillators except that a_i and $a^{\dagger}_j$ are replaced by a'_i and $a'^{\dagger}_j$. However, there are two important differences. First, due to the constraint $\sum_{j=1}^{N} a'^{\dagger}_j = 0$, some of the eigen-wavefunctions are in fact zero or redundant. For example, $\psi_{\text{rel},\{1,0,...,0\}}$ is actually zero and $\psi_{\text{rel},\{1,1,0,...,0\}}$ can be reduced to $\psi_{\mathrm{rel},\{2,0,\dots,0\}}$. Second, we can see from Eq. (5) that although the wavefunctions with different total energies are orthogonal to each other, those with the same energies, or degenerate states, are not.

To see how many wavefunctions in (16) for a fixed $\sum_{i=1}^{N} n_i = n$ are really independent, let us consider N non-interacting Bose harmonic oscillators. Suppose we know that the degeneracy of its nth energy level is D_n. It is easy to see that if Ψ_{n-1} is one of the D_{n-1} eigenfunctions of the $(n-1)$th level then $\sum_{i=1}^{N} a_i^\dagger \Psi_{n-1}$ will also be an eigenfunction of its nth level. But after we replace $a_i^\dagger$ by $a_i'^\dagger$, all these D_{n-1} states vanish. Therefore, the number of independent states or the degeneracy of nth energy level of $H_{\mathrm{rel}}(\omega)$ is:

$$D_{\mathrm{rel},n} = D_n - D_{n-1} \qquad (n = 1, 2,\dots) \tag{18}$$

We can immediately see that $D_{\mathrm{rel},1} = 0$. These $D_{\mathrm{rel},n}$ independent states can be orthogonalized by choosing the appropriate linear combinations of the original D_n states in (16). This can be done in principle for any nth level, but the algebra gets messy as n increases. Here we only list the independent symmetrized $\psi_{\mathrm{rel},n}$ up to $n = 4$.

$$\begin{aligned}
\psi_{\mathrm{rel},2} &= \sum_{i=1}^{N} a_i'^{\dagger 2}\psi_{\mathrm{rel},0} = 2\left(\omega \sum_{i=1}^{N} x_i'^2 - \frac{N-1}{2}\right)\psi_{\mathrm{rel},0}\\
\psi_{\mathrm{rel},3} &= \sum_{i=1}^{N} a_i'^{\dagger 3}\psi_{\mathrm{rel},0} = (2\omega)^{3/2} \sum_{i=1}^{N} x_i'^3 \psi_{\mathrm{rel},0}\\
\psi_{\mathrm{rel},4a} &= \left(\sum_{i=1}^{N} a_i'^{\dagger 2}\right)^2 \psi_{\mathrm{rel},0}\\
&= \left[4\omega^2\left(\sum_{i=1}^{N} x_i'^2\right)^2 - 4(N+1)\,\omega \sum_{i=1}^{N} x_i'^2 + (N+1)(N-1)\right]\psi_{\mathrm{rel},0}\\
\psi_{\mathrm{rel},4b} &= \sum_{i=1}^{N} a_i'^{\dagger 4}\,\psi_{\mathrm{rel},0}\\
&= \left[4\omega^2 \sum_{i=1}^{N} x_i'^4 - \frac{12(N-1)\,\omega}{N} \sum_{i=1}^{N} x_i'^2 + \frac{3(N-1)^2}{N}\right]\psi_{\mathrm{rel},0}
\end{aligned} \tag{19}$$

Note that $\psi_{\mathrm{rel},4a}$ and $\psi_{\mathrm{rel},4b}$ are independent but not orthogonal. All $\psi_{\mathrm{rel},4}$'s can be written as their linear combinations.

Combining Eq. (9) and Eq. (17), the (n, m) energy level of the 1d harmonic interaction model is

$$E_{n,m} = \left(n + \frac{N-1}{2}\right)\omega + m + \frac{1}{2} \qquad (n = 0, 2, 3,\dots;\ m = 0, 1, 2,\dots) \tag{20}$$

The spectrum is specified by two quantum numbers: relative excitation n and center of mass excitation m. The degeneracy of level (n, m) is $D_{n,m} = D_{\mathrm{rel},n}$. For $\omega > 1$ or $K > 0$ case, the spectrum indicates a band structure. Within the nth energy band, there are n sub-energy levels ($n \neq 0$). The spacings between them are 2Δ, Δ,..., Δ, where $\Delta = \omega - 1$. The states without center of mass excitation are the highest (lowest) ones in the energy band for $\omega > 1$ ($\omega < 1$) except at the $n = 1$ level. This is one explicit example of how the interaction causes energy level breaks into bands.

Now, we are ready to calculate the free energy of the harmonic interaction model at finite temperature T. The N-body partition function is

$$\begin{aligned}
Z(\beta; \omega) &= \sum_{n,m} D_{n,m}\, e^{-\beta E_{n,m}} \\
&= \left(\sum_n D_{\mathrm{rel},n}\, e^{-\beta n\omega}\right)\left(\sum_m e^{-\beta m}\right) e^{-\beta(\frac{N-1}{2}\omega + \frac{1}{2})} \\
&= \left(\sum_n (D_n - D_{n-1})\, e^{-\beta n\omega}\right)\left(\sum_m e^{-\beta m}\right) e^{-\beta(\frac{N-1}{2}\omega + \frac{1}{2})} \\
&= \frac{\sinh \frac{\beta\omega}{2}}{\sinh \frac{\beta}{2}} Z_0(\beta; \omega)
\end{aligned} \tag{21}$$

where $Z_0(\beta; \omega) = (\sum_n D(n)\, e^{-\beta n\omega})\, e^{-\beta N\omega/2}$ is the partition function for N non-interacting Bose 1-d harmonic oscillators with frequency ω and $\beta = 1/kT$. The free energy follows,

$$F(\beta; \omega) = -\frac{1}{\beta} \ln Z(\beta; \omega) = F_0(\beta; \omega) - \frac{1}{\beta} \ln \left(\frac{\sinh \frac{\beta\omega}{2}}{\sinh \frac{\beta}{2}}\right) \tag{22}$$

where $F_0(\beta; \omega)$ is the free energy for non-interacting Bosons.

The harmonic interaction model in d dimensions can be solved following the same line as the 1-d case. The energy (n, m) level of Eq. (1) now is given by

$$E^{(d)}_{n,m} = \left[n + \frac{(N-1)\,d}{2}\right]\omega + m + \frac{d}{2} \qquad (n = 0, 2, 3, \ldots;\ m = 0, 1, 2, \ldots) \tag{23}$$

The degeneracy of the (n, m) state is

$$D^{(d)}_{n,m} = D^{(d)}_{\mathrm{rel},n}\, D^{(d)}_{1,m} \tag{24}$$

where $D^{(d)}_{\text{rel},n}$ is the degeneracy of the nth relative energy level and $D^{(d)}_{1,m}$ is the degeneracy of the mth level of a single d dimension harmonic oscillator. $D^{(d)}_{\text{rel},n}$ can be determined by

$$D^{(d)}_n = \sum_{m=0}^{n} D^{(d)}_{n-m,m} = \sum_{m=0}^{n} D^{(d)}_{\text{rel},n-m}\, D^{(d)}_{1,m} \qquad (n = 0, 1, \ldots) \tag{25}$$

where $D^{(d)}_n$ is the degeneracy of the nth level of N non-interacting Bose d-dimension harmonic oscillators with frequency ω. Equation (25) comes from the fact that the total number of states in the nth band is equal to those of nth level when $\omega = 1$ (non-interacting). In the special case $d = 1$, $D^{(1)}_{1,m} = 1$ for all m and Eq. (25) can be shown equivalent to Eq. (18).

From Eq. (24), we know that the partition function of d dimensions harmonic interaction model is given by

$$Z^{(d)}(\beta;\omega) = Z^{(d)}_{\text{rel}}(\beta;\omega)\, z^{(d)}_1(\beta;1) \tag{26}$$

where $Z^{(d)}_{\text{rel}}(\beta;\omega)$ is the contribution from the relative part and $z^{(d)}_1(\beta;1)$ is from the center of mass part. On the other hand, for N non-interacting Bose harmonic oscillators with frequency ω, its partition function $Z^{(d)}_0(\beta;\omega) = \sum_n D^{(d)}_n e^{-(n+Nd/2)\beta\omega}$. From Eqs. (23) and (25), it can be written as

$$Z^{(d)}_0(\beta;\omega) = Z^{(d)}_{\text{rel}}(\beta;\omega)\, z^{(d)}_1(\beta;\omega) \tag{27}$$

Dividing Eq. (26) by Eq. (27), we obtain

$$Z^{(d)}(\beta;\omega) = Z^{(d)}_0(\beta;\omega)\,\frac{z^{(d)}_1(\beta;1)}{z^{(d)}_1(\beta;\omega)} \tag{28}$$

The free energy is a direct generalization of Eq. (22)

$$F^{(d)}(\beta;\omega) = F^{(d)}_0(\beta;\omega) - \frac{d}{\beta}\ln\left(\frac{\sinh\frac{\beta\omega}{2}}{\sinh\frac{\beta}{2}}\right) \tag{29}$$

where we have made use of $z^{(d)}_1(\beta;\omega) = 2^d/\sinh^d\frac{\beta\omega}{2}$.

To make a comparison with the mean-field solution, we calculate the one-body density function in 3-d by integration of the normalized ground state wavefunction

$$\Psi_0(\mathbf{r}_1,\ldots,\mathbf{r}_N) = \frac{\omega^{3(N-1)/4}}{\pi^{3N/4}}\exp\left[-\frac{\omega}{2}\sum_{i=1}^{N} r_i^2 + \frac{\epsilon}{2}\left(\sum_{i=1}^{N}\mathbf{r}_i\right)^2\right] \tag{30}$$

The one-body density function takes the form

$$\rho_1(\mathbf{r}) = N\left[\frac{\omega}{(1+\epsilon)\,\pi}\right]^{\frac{3}{2}} e^{-\frac{\omega}{1+\epsilon}r^2} \tag{31}$$

where $\epsilon = (\omega - 1)/N$. On the other hand, the mean-field equation for interaction ϕ is

$$-\tfrac{1}{2}\nabla^2\psi(\mathbf{r}) + u(\mathbf{r})\,\psi(\mathbf{r}) + \psi(\mathbf{r})\int N\phi(\mathbf{r}-\mathbf{r}')\,\psi^2(\mathbf{r}')\,d\mathbf{r}' = \mu\psi(\mathbf{r}) \tag{32}$$

where $\psi(\mathbf{r}) = \sqrt{\rho_1(\mathbf{r})/N}$.

If we take $\phi(\mathbf{r}-\mathbf{r}') = K(\mathbf{r}-\mathbf{r}')^2$, Eq. (32) has the solution

$$\psi(\mathbf{r}) = \left(\frac{\omega}{\pi}\right)^{\frac{3}{4}} e^{-\frac{1}{2}\omega r^2} \tag{33}$$

with $\mu = \frac{3}{2}\,\omega$.

Comparing the mean-field solution Eq. (33) with the exact solution Eq. (31), one can see that if ω is fixed as constant and $\epsilon \to 0$ (NK is kept constant and $N \to \infty$) then the mean-field solution is recovered. At finite N, the leading correction to the mean-field solution is of the order of $\epsilon = (\omega - 1)/N$. The density profile becomes less narrow for attractive interaction ($K > 0$) and less broad for repulsive interaction ($K < 0$) than the mean field solution. In fact, a rigorous derivation of the mean-field approximation from the basic Hamiltonian with genuine particle interaction has been established recently in ref. 10.

3. COMPARISON WITH PSEUDO-POTENTIAL SYSTEM

So far, the harmonic interaction model has merely served as a "toy model." In all realized BEC experiments up to now, the interactions between atoms are all of short-range nature. One might doubt the validity of using the long-range harmonic interaction model as a reference model to study the real BEC. However, in both situations, the particles have been confined in a finite region by the harmonic trap, so the difference between long-range and short-range interaction is not so significant. In our model, there is only one controlling parameter ω. If we can find a way to obtain the effective ω for the short-range interaction system, our model under this effective ω may give some relevant properties of the real system.

The simplest way one can think of is via a variational principle. If we use Eq. (30) as a variational ansartz for the ground state wavefunction of the real system, we can obtain the optimal ω by minimizing the expectation value of energy. The Hamiltonian of the real system is

$$H_{\text{real}} = \frac{1}{2} \sum_{i=1}^{N} (p_i^2 + r_i^2) + \sum_{i<j} \phi(\mathbf{r}_i - \mathbf{r}_j) \tag{34}$$

where $\phi(\mathbf{r}_i - \mathbf{r}_j)$ can be any kind of interaction.

The expectation value of Eq. (34) can be calculated as

$$\begin{aligned} E_{\text{real}}(\omega) &= \langle \Psi_0 | H_{\text{real}} | \Psi_0 \rangle \\ &= \langle \Psi_0 | H_{\text{har}} | \Psi_0 \rangle + \langle \Psi_0 | H_{\text{real}} - H_{\text{har}} | \Psi_0 \rangle \\ &= \frac{3}{2} + \frac{3}{2} (N-1)\, \omega \\ &\quad + \frac{N(N-1)}{2} \int \rho_2(\mathbf{r}, \mathbf{r}')[\phi(\mathbf{r}-\mathbf{r}') - K(\mathbf{r}-\mathbf{r}')^2]\, d\mathbf{r}\, d\mathbf{r}' \\ &= \frac{3}{2} + \frac{N(N-1)}{2} \langle \phi(\sqrt{2}\,\mathbf{r}) \rangle_\omega + \frac{3(N-1)}{4} \left(\omega + \frac{1}{\omega} \right) \end{aligned} \tag{35}$$

where $\langle \phi(\sqrt{2}\,\mathbf{r}) \rangle_\omega = \langle \phi_0(\mathbf{r};\omega) | \phi(\sqrt{2}\,\mathbf{r}) | \phi_0(\mathbf{r};\omega) \rangle$ and $\phi_0(\mathbf{r};\omega) = (\frac{\omega}{\pi})^{3/4} e^{-\frac{1}{2}\omega r^2}$.

The optimal ω can be determined by $\frac{dE_{\text{real}}(\omega)}{d\omega} = 0$, that is

$$N \frac{d}{d\omega} \langle \phi(\sqrt{2}\,\mathbf{r}) \rangle_\omega + \frac{3}{2} \left(1 - \frac{1}{\omega^2} \right) = 0 \tag{36}$$

To make a comparison, we use the pseudo-potential $\phi(\mathbf{r}) = g\delta^{(3)}(\mathbf{r})$, where $g = 4\pi a$ and a is the scattering length. Now, $\langle \phi(\sqrt{2}\,\mathbf{r}) \rangle_\omega = g(\frac{\omega}{2\pi})^{3/2}$ and Eq. (36) becomes

$$\sqrt{\frac{2}{\pi}}\, N a \omega^{1/2} + 1 - \frac{1}{\omega^2} = 0 \tag{37}$$

where Na is a dimensionless quantity often used in pseudo-potential GP equation to describe the interaction strength between atoms. From Eq. (37), we can solve for ω as a function of Na. Substituting (37) back in (35), we have

$$E_{\text{real}} = \frac{3}{2} + \frac{1}{4} (N-1) \left(\omega + \frac{5}{\omega} \right) \tag{38}$$

In the so-called Thomas–Fermi limit, $Na \gg 1$, Eq. (37) gives $\omega = (\sqrt{\frac{2}{\pi}}\, Na)^{2/5}$ and Eq. (38) becomes $E_{\text{real}} = \frac{5}{4}\,(\frac{2}{\pi})^{1/5}(Na)^{2/5}(N-1)$ which can be compared with the GP equation's Thomas–Fermi limit $E = \frac{5}{14}\,(15Na)^{2/5}N$.

We observe that Eq. (37) has solutions for ω only when $Na \geqslant Na_{\text{critical}} = -\sqrt{\frac{\pi}{2}}\,4/5^{5/4} = -0.671$ and they all correspond to the minimum of Eq. (35). The existence of the negative lower bound for Na corresponds to the critical collapse value of the attractive interaction. The collapse of attractive interactions has been observed in the tunable Na BEC system and our critical value is quite close to $Na_{\text{critical}} = -0.575$ given by the GP equation.

We calculate the density profile of Eq. (31) under the ω determined by Eq. (37). For repulsive interaction ($a > 0$), it gives $\omega < 1$, which corresponds to a wider spread of density than in the non-interacting case. We find good agreement with the numerical solution of GP equation when Na is small. Because the density given by our model is intrinsically Gaussian, we are not able to give the nonlinear behavior of the GP equation when Na is large.

If we take ω determined by Eq. (37) as effective parameter in the harmonic interaction model for repulsive interaction ($a > 0$), the energy spectrum moves toward the ground state. The number of states below a fixed excitation energy has been increased. The excitation spectrum of the pseudo-potential system has been solved numerically by Bogoliubov-type equation. It follows the same trend except that more energy levels are broken than in the harmonic interaction model.

For the harmonic interaction model, the thermodynamic limit is taken as $N \to \infty$, and NK is fixed, or ω is fixed. The free energy (29) then will just be the one for N non-interacting Bose harmonic oscillators with frequency ω. In 3-dimensions, the BEC transition temperature is just

$$k_B T_c = \omega \left[\frac{N}{\zeta(3)} \right]^{\frac{1}{3}} \tag{39}$$

where $\zeta(x)$ is Riemann zeta-function. T_c is lower than in the non-interacting case for $\omega < 1$. Meanwhile, study of the thermodynamic properties of the pseudo-potential system also suggests that T_c becomes lower for $a > 0$.

We also observe that the type of exact solution for the harmonic interaction model which represents vortex states(11, 12)

$$\begin{aligned} \Psi_v &= \prod_{j=1}^{N} a'^{\dagger}_{j,\pm}\ {}^{\kappa}\Psi_0 \\ &= \prod_{j=1}^{N} (x'_j \pm i y'_j)\ {}^{\kappa}\Psi_0 \end{aligned} \tag{40}$$

is of the form of Eq. (16) in 3-d, where $a'_{j,\pm} = (a'_{j,x} \pm i a'_{j,y})/\sqrt{2}$. Therefore it is the exact eigenfunction of the 3-d isotropic harmonic interaction model, whereas the wavefunction with the primes omitted in Eq. (40) is not when $K \neq 0$. Interestingly, Eq. (40) is exactly the same form that Wilkin *et al.*[13] proposed for the vortex state for the pseudo-potential system. Furthermore, the critical rotational frequency for the onset of vortices in harmonic interaction is exactly ω. For repulsive interaction ($a > 0$), it is smaller than 1 which is the critical frequency for a non-interacting system. This again agrees with the prediction from the pseudo-potential GP equation.

As we have seen above, the harmonic interaction model shares many properties with the pseudo-potential system in spite of their different nature. This can be explained as follows. For the long range harmonic interaction, due to the trap confinement, the two body correlation function in the ground state can be shown to be the short range Gaussian shape function. The harmonic interaction is weighted by the correlation function therefore only its short range part matters. The short range repulsive interaction will resemble the $K < 0$ harmonic interaction because it has negative curvature near $\mathbf{r}_{ij} = 0$. For the same reason, the short range attractive interaction corresponds to $K > 0$.

4. DISCUSSIONS

We have shown in Section 2 that quite a few properties of the harmonic interaction model are well understood due to its decoupling into relative and center of mass part. One of the most important features of the harmonic interaction model is that it can be solved for any N so that we can see finite N effects clearly.

We compared the harmonic interaction model and pseudo-potential interaction model from ground state density, spectrum, transition temperature and vortex state viewpoints. All these suggest similarities between two models. The variational principle we used in Section 3 to find an effective ω for corresponding a is still very primitive and only appropriate for the ground state, strictly speaking. To really draw a rigorous bridge between the two models, more sophisticated methods such as those employing unitary transformations may be employed.

ACKNOWLEDGMENTS

The author would like to thank Prof. Percus for advices and two anonymous referees for important comments.

REFERENCES

1. M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, *Science* **269**:198 (1995).
2. K. B. Davis, M. O. Mewes, M. R. Andrews, N. J. Van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, *Phys. Rev. Lett.* **75**:3969 (1995).
3. M. Girardeau, *J. Math. Phys.* **1**:516 (1960).
4. E. Lieb and W. Liniger, *Phys. Rev.* **130**:1605 (1963).
5. L. Dukelsky and P. Schuck, *Phys. Rev. Lett.* **86**:4207 (2001).
6. L. Dukelsky, C. Esebbag, and P. Schuck, *Phys. Rev. Lett.* **87**:066403 (2001).
7. M. Katzper and J. K. Percus, unpublished (1967).
8. N. K. Wilkin, J. M. F. Gunn, and R. A. Smith, *Phys. Rev. Lett.* **80**:2265 (1998).
9. M. A. Zaluska-Kotur, M. Gajda, A. Orlowski, and J. Mostowski, *Phys. Rev. A.* **61**: 033613 (2000).
10. E. Lieb and R. Seiringer, *Phys. Rev. Lett.* **88**:170409 (2002).
11. M. R. Matthews *et al.*, *Phys. Rev. Lett.* **83**:2498 (1999).
12. K. W. Madison, F. Chevy, W. Wohlleben, and J. Dalibard, *Phys. Rev. Lett.* **84**:806 (2000).
13. N. K. Wilkin and J. M. F. Gunn, *Phys. Rev. Lett.* **84**:6 (2000).